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INEQUALITIES FOR THE INTEGRAL MEANS OF HOLOMORPHIC FUNCTIONS AND THEIR DERIVATIVES IN THE UNIT BALL OF C^n

SHI JI-HUAI

ABSTRACT. In this paper, the following two inequalities are proved:

$$\int_0^1 (1-r)^{a|\alpha|+b} M_p^a(r, D^\alpha f)\, dr \le K \int_0^1 (1-r)^b M_p^a(r, f)\, dr,$$

$$\int_0^1 (1-r)^b M_p^a(r, f)\, dr$$
$$\le K \left\{ \sum_{|\alpha|\le m-1} |(D^\alpha f)(0)|^a + \sum_{|\alpha|=m} \int_0^1 (1-r)^{am+b} M_p^a(r, D^\alpha f)\, dr \right\}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is multi-index, $0 < p < \infty$, $0 < a < \infty$ and $-1 < b < \infty$. These are a generalization of some classical results of Hardy and Littlewood. Using these two inequalities, we generalize a theorem in [9]. The methods used in the proof of Theorem 1 lead us to obtain Theorem 7, which enables us to generalize some theorems about the pluriharmonic conjugates in [8] and [2].

1. INTRODUCTION

Let B denote the unit ball of complex vector space C^n. The letter ν stands for the Lebesgue measure on C^n, so normalized that $\nu(B) = 1$, while σ is the surface measure on the boundary ∂B of B normalized so that $\sigma(\partial B) = 1$.

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in C^n, denote the inner product of z, w by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$, $|z|^2 = \langle z, z\rangle$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be multi-index, α_j being nonnegative integers, we will write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!, \qquad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

By $H(B)$ we denote the class of all functions holomorphic in B. For $f \in H(B)$, denote

$$(D^\alpha f)(z) = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(z).$$

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The integral means $M_p(r, f)$ of f, $0 < p \leq \infty$, are defined by

$$M_p(r, f) = \left\{ \int_{\partial B} |f(r\zeta)|^p \, d\sigma(\zeta) \right\}^{1/p}, \qquad 0 < p < \infty,$$
$$M_\infty(r, f) = \sup\{|f(r\zeta)| : \zeta \in \partial B\}.$$

The main results of this paper are the following two theorems.

Theorem 1. *Let $f \in H(B)$, $0 < p < \infty$, $-1 < b < \infty$ and $0 < a < \infty$. Then for any $\alpha = (\alpha_1, \dots, \alpha_n)$, there exists a positive constant K independent of f, such that*

$$\int_0^1 (1-r)^{a|\alpha|+b} M_p^a(r, D^\alpha f)\, dr \leq K \int_0^1 (1-r)^b M_p^a(r, f)\, dr.$$

Theorem 2. *Let $f \in H(B)$, $0 < p < \infty$, $-1 < b < \infty$ and $0 < a < \infty$. Then for any positive integer m, we have*

$$\int_0^1 (1-r)^b M_p^a(r, f)\, dr$$
$$\leq K \left\{ \sum_{|\alpha| \leq m-1} |(D^\alpha f)(0)|^a + \sum_{|\alpha|=m} \int_0^1 (1-r)^{am+b} M_p^a(r, D^\alpha f)\, dr \right\}.$$

Here, and later, K always denotes a positive constant, not necessarily the same at each occurrence; it is independent of f.

These two theorems generalize the classical results of Hardy and Littlewood (see [3, p. 81]) to the unit ball of C^n. The proofs of Theorems 1 and 2 will be given in §§2 and 3 respectively.

In §4, we will give an application of Theorems 1 and 2. More recently, Zhu [9] proved the following theorem:

Theorem A. *Let m be a positive integer and $f \in H(B)$, then $f \in L^p(d\nu)$, $1 \leq p < \infty$, if and only if the functions $(1-|z|^2)^m (D^\alpha f)(z)$ with $|\alpha| = m$ are in $L^p(d\nu)$.*

Using Theorems 1 and 2, we will give a new proof of Theorem A, moreover, we will prove that it also holds for $0 < p < 1$.

A continuous real function u on B is pluriharmonic if for every holomorphic mapping γ of U into B, U is the unit disc, $u \circ \gamma$ is harmonic in U. It is well known that every pluriharmonic function on B is the real part of a holomorphic function. Let u be pluriharmonic on B, then $u = \operatorname{Re} f$, where $f = u + iv$ is holomorphic in B and v is called the pluriharmonic conjugate of u.

Stoll proved the following two theorems in [8]:

Theorem B. *Let $f = u + iv$ be holomorphic in B, if*

$$M_p(r, u) = O((1-r)^{-\alpha}), \qquad 1 \leq p \leq \infty, \ \alpha > 0,$$

then

$$M_p(r, v) = O((1-r)^{-\alpha}).$$

Theorem C. *Let $f = u + iv$ be holomorphic in B with $f(0)$ real. Then for $-1 < b < \infty$, $1 \le p \le \infty$, $0 < a < \infty$,*

$$\int_0^1 (1-r)^b M_p^a(r, v)\, dr \le K \int_0^1 (1-r)^b M_p^a(r, u)\, dr.$$

We have generalized the Theorems B and C to bounded symmetric domain of C^n in [7], but the problem which is still unsolved is the case $0 < p < 1$.

In §5, applying the approach used in the proof of Theorem 1, we first prove the following inequality:

Theorem 7. *Let $f = u + iv$ be holomorphic in B and $0 < p < 1$, $s \ge 0$, then*

$$|f(z)|^p \le K\left\{|v(0)|^p + \int_B |u(w)|^p \frac{(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+1}}\, d\nu(w)\right\}, \qquad z \in B.$$

When $n = 1$, this theorem has been proved in [5], we generalize it to the unit ball of C^n.

Using Theorem 7, we will prove that Theorems B and C hold in the case $0 < p < 1$.

In [2] Chen introduced the spaces $h\wedge(\alpha, p, q)$ on the bounded symmetric domain D of C^n as follows: $h\wedge(\alpha, p, q)$, $0 < p \le \infty$, $0 < q \le \infty$, $\alpha > 0$, is the set of pluriharmonic functions on D with $N_{p,q,\alpha}(u) < \infty$, where

$$N_{p,q,\alpha}(u) = \left\{\int_0^1 (1-r)^{nq\alpha-1} M_p^q(r, u)\, dr\right\}^{1/q}, \qquad 0 < q < \infty,$$

$$N_{p,\infty,\alpha}(u) = \sup_{0 \le r < 1}\{(1-r)^{n\alpha} M_p(r, u)\},$$

and he proved that $h\wedge(\alpha, p, q)$ is self-conjugate class for $1 \le p \le \infty$, $0 < q \le \infty$ and $\alpha > 0$, that is, if $u \in h\wedge(\alpha, p, q)$, then the pluriharmonic conjugate $v \in h\wedge(\alpha, p, q)$. The problem which is also unsolved is the case $0 < p < 1$. As a simple application of the generalized Theorem C, we will claim that $h\wedge(\alpha, p, q)$ is self-conjugate class for $0 < p \le \infty$, $0 < q \le \infty$ and $\alpha > 0$ if $D = B$.

2. The proof of Theorem 1

We first prove the following lemmas.

Lemma 1. *Let $f \in H(B)$ and $0 < p < \infty$, then*

$$\text{(1)} \qquad \text{(a)} \quad |f(a)|^p \le \frac{K}{(1-|a|)^{n+1}} \int_B |f(z)|^p\, d\nu(z)$$

for any $a \in B$.

$$\text{(2)} \qquad \text{(b)} \quad |f(\tfrac{1}{2}\zeta)|^p \le K \int_B |f(z)|^p (1-|z|^2)^s\, d\nu(z)$$

for any $\zeta \in \partial B$ and $s \ge 0$.

Proof. (a) Since $f \in H(B)$, $|f|^p$ is subharmonic for $0 < p < \infty$, and

$$|f(0)|^p \le \int_B |f(w)|^p \, d\nu(w). \tag{3}$$

Let φ_a be a holomorphic automorphism of B with $\varphi_a(0) = a$ [6, p. 25]. Replacing f by $f \circ \varphi_a$ in (3) gives

$$|f(a)|^p \le \int_B |f \circ \varphi_a(w)|^p \, d\nu(w).$$

Let $z = \varphi_a(w)$, then $w = \varphi_a(z)$, $d\nu(w) = ((1 - |a|^2)|1 - \langle a, z\rangle|^{-2})^{n+1} \, d\nu(z)$ [6, p. 28] and

$$|f(a)|^p \le \int_B |f(z)|^p \left\{ \frac{1 - |a|^2}{|1 - \langle a, z\rangle|^2} \right\}^{n+1} d\nu(z) \le \frac{K}{(1 - |a|)^{n+1}} \int_B |f(z)|^p \, d\nu(z).$$

(1) follows.

(b) Let $f_r(z) = f(rz)$ for $0 < r < 1$, then

$$|f(ra)|^p \le \frac{K}{(1 - |a|)^{n+1}} \int_B |f(rz)|^p \, d\nu(z) \tag{4}$$

by (1). Taking $r = \frac{1}{\sqrt{2}}$, $a = \frac{1}{\sqrt{2}}\zeta$, $\zeta \in \partial B$ in (4) gives

$$|f(\tfrac{1}{2}\zeta)|^p \le K \int_B \left| f\left(\frac{1}{\sqrt{2}} z \right) \right|^p d\nu(z) \le K \int_{\frac{1}{\sqrt{2}}B} |f(z)|^p \, d\nu(z) \tag{5}$$

where $\frac{1}{\sqrt{2}}B = \{z \in C^n : |z| < \frac{1}{\sqrt{2}}\}$.

On the other hand, by the formula in [6, 1.4.3],

$$\begin{aligned} \int_B |f(z)|^p (1 - |z|^2)^s \, d\nu(z) &= 2n \int_0^1 t^{2n-1} \, dt \int_{\partial B} |f(t\zeta)|^p (1 - t^2)^s \, d\sigma(\zeta) \\ &\ge K \int_{\frac{1}{\sqrt{2}}B} |f(z)|^p \, d\nu(z). \end{aligned} \tag{6}$$

Now (2) follows from (5) and (6).

Lemma 2. *Let $f \in H(B)$ and $0 < p < \infty$, $s \ge 0$, $n + s + 1 \ge p$, then*

$$|(\nabla f)(z)|^p \le K \int_B |f(w)|^p \frac{(1 - |w|^2)^s}{|1 - \langle z, w\rangle|^{n+s+p+1}} \, d\nu(w) \tag{7}$$

for $z \in B$, where $(\nabla f)(z) = (\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$ is an analytic gradient.

Proof. By the Taylor expansion of f, we have

$$\int_{\partial B} f(r\zeta)\overline{\zeta}^\alpha \, d\sigma(\zeta) = \frac{(D^\alpha f)(0)}{\alpha!} \omega_\alpha r^{|\alpha|} \tag{8}$$

where $\omega_\alpha = \int_{\partial B} |\zeta^\alpha|^2 \, d\sigma(\zeta) = \frac{(n-1)!\,\alpha!}{(n+|\alpha|-1)!}$ [6, p. 16]. Taking $\alpha = (0, \dots, 1, \dots, 0)$, the kth coordinate is 1, and the rest are 0, $r = \frac{1}{2}$ in (8) and using Lemma 1(b) yield

$$\left| \frac{\partial f}{\partial z_k}(0) \right|^p \le K \sup_{\zeta \in \partial B} |f(\tfrac{1}{2}\zeta)|^p \le K \int_B |f(z)|^p (1 - |z|^2)^s \, d\nu(z), \qquad k = 1, \dots, n.$$

By the elementary inequality

$$(a+b)^p \le \begin{cases} a^p + b^p, & 0 < p < 1, \\ 2^{b-1}(a^p + b^p), & p \ge 1, \end{cases} \qquad a > 0, b > 0,$$

we have

$$|(\nabla f)(0)|^p \le K \int_B |f(z)|^p (1-|z|^2)^s \, d\nu(z). \tag{9}$$

Let $F = f \circ \varphi_z$, then $(\nabla F)(0) = (\nabla f)(z)\varphi_z'(0)$, where φ_z' is the derivative of φ_z, namely, the Jacobian matrix of φ_z. A direct computation gives [6, p. 48]

$$\varphi_z'(0) = \frac{\delta}{1+\delta} z'\overline{z} - \delta I, \qquad \delta^2 = 1 - |z|^2,$$

where z' is a column vector, the transpose of z, I the identity matrix of order n, and

$$\varphi_z'(0)(\varphi_z'(0))^* = \delta^2(I - z'\overline{z}),$$

where $(\varphi_z'(0))^*$ is the conjugate transpose of the matrix $\varphi_z'(0)$. Thus

$$\begin{aligned} |(\nabla F)(0)|^2 &= \delta^2 (\nabla f)(z)(I - z'\overline{z})((\nabla f)(z))^* \\ &= \delta^2\{|(\nabla f)(z)|^2 - |(\nabla f)(z)z'|^2\} \ge \delta^2 |(\nabla f)(z)|^2(1-|z|^2), \end{aligned}$$

that is

$$|(\nabla F)(0)| \ge (1-|z|^2)|(\nabla f)(z)|. \tag{10}$$

Replace f by F in (9), and change the variable w by $\tau = \varphi_z(w)$, and note the identity

$$1 - |\varphi_z(\tau)|^2 = \frac{(1-|z|^2)(1-|\tau|^2)}{|1-\langle z, \tau\rangle|^2} \qquad [6, \text{p. } 26],$$

we have

$$\begin{aligned} |(\nabla F)(0)|^p &\le K \int_B |(f \circ \varphi_z)(w)|^p (1-|w|^2)^s \, d\nu(w) \\ &= K \int_B |f(\tau)|^p \frac{(1-|z|^2)^s (1-|\tau|^2)^s}{|1-\langle z, \tau\rangle|^{2s}} \left\{ \frac{1-|z|^2}{|1-\langle z, \tau\rangle|^2} \right\}^{n+1} d\nu(\tau) \\ &= K \int_B |f(\tau)|^p \frac{(1-|\tau|^2)^s (1-|z|^2)^{n+s+1}}{|1-\langle z, \tau\rangle|^{2(n+s+1)}} \, d\nu(\tau). \end{aligned}$$

Now the inequality (10) implies

$$\begin{aligned} |(\nabla f)(z)|^p &\le K \int_B |f(\tau)|^p \frac{(1-|\tau|^2)^s (1-|z|^2)^{n+s+1-p}}{|1-\langle z, \tau\rangle|^{2(n+s+1)}} \, d\nu(\tau) \\ &\le K \int_B |f(\tau)|^p \frac{(1-|\tau|^2)^s}{|1-\langle z, \tau\rangle|^{n+s+p+1}} \, d\nu(\tau). \end{aligned}$$

The lemma is proved.

Lemma 3. *Let* $f \in H(B)$ *and* $0 < q \leq p < \infty$, $s \geq 0$, $n+s+1 \geq q$, *then*

$$M_p^q\left(r, \frac{\partial f}{\partial z_k}\right) \leq K \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+q+1}} M_p^q(\rho, f)\, d\rho, \tag{11}$$

for $0 < r < 1$ *and* $k = 1, \ldots, n$.

Proof. By Lemma 2,

$$\left|\frac{\partial f}{\partial z_k}(r\zeta)\right|^q \leq |(\nabla f)(r\zeta)|^q \leq K \int_B |f(w)|^q \frac{(1-|w|^2)^s}{|1-\langle r\zeta, w\rangle|^{n+s+q+1}}\, d\nu(w)$$

for any $0 < r < 1$ and $\zeta \in \partial B$. For given $\zeta \in \partial B$, there exist $\eta_2, \ldots, \eta_n \in \partial B$, such that

$$V = \begin{pmatrix} \zeta \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$$

is a unitary matrix. Let $w = \tau V$, then

$$\langle \zeta, w\rangle = \langle \zeta, \tau V\rangle = \zeta(\zeta^*, \eta_2^*, \ldots, \eta_n^*)\tau^* = (1, 0, \ldots, 0)\tau^* = \overline{\tau}_1.$$

Thus

$$\left|\frac{\partial f}{\partial z_k}(r\zeta)\right|^q \leq K \int_B |f(\tau V)|^q \frac{(1-|\tau|^2)^s}{|1-r\overline{\tau}_1|^{n+s+q+1}}\, d\nu(\tau). \tag{12}$$

Let $\beta = p/q \geq 1$, using (12) and Minkowski's inequality in infinite form imply

(13)

$$\begin{aligned} M_p^q\left(r, \frac{\partial f}{\partial z_k}\right) &= \left\{\int_{\partial B} \left|\frac{\partial f}{\partial z_k}(r\zeta)\right|^{q\beta} d\sigma(\zeta)\right\}^{1/\beta} \\ &\leq K \left\{\int_{\partial B} \left\{\int_B |f(\tau V)|^q \frac{(1-|\tau|^2)^s}{|1-r\overline{\tau}_1|^{n+s+q+1}}\, d\nu(\tau)\right\}^\beta d\sigma(\zeta)\right\}^{1/\beta} \\ &\leq K \int_B \frac{(1-|\tau|^2)^s}{|1-r\overline{\tau}_1|^{n+s+q+1}} \left\{\int_{\partial B} |f(\tau V)|^p\, d\sigma(\zeta)\right\}^{q/p} d\nu(\tau) \\ &\leq K \int_0^1 \rho^{2n-1}(1-\rho^2)^s\, d\rho \int_{\partial B} \frac{d\sigma(\eta)}{|1-r\rho\overline{\eta}_1|^{n+s+q+1}} \\ &\quad \times \left\{\int_{\partial B} |f(\rho\eta V)|^p\, d\sigma(\zeta)\right\}^{q/p}. \end{aligned}$$

By the formula of [6, 1.4.7], for any $\eta \in \partial B$,

(14)

$$\int_{\partial B} |f(\rho\eta V)|^p\, d\sigma(\zeta) \leq \int_{\mathscr{U}} |f(\rho\eta U)|^p\, dU = \int_{\partial B} |f(\rho\zeta)|^p\, d\sigma(\zeta) = M_p^p(\rho, f),$$

where $\mathscr{U}$ is the unitary group of order n and dU the Haar measure on $\mathscr{U}$. Combining (13) and (14) yield

$$\begin{aligned} M_p^q\left(r, \frac{\partial f}{\partial z_k}\right) &\le K \int_0^1 (1-\rho)^s M_p^q(\rho, f)\, d\rho \int_{\partial B} \frac{d\sigma(\eta)}{|1 - r\rho\overline{\eta}_1|^{n+s+q+1}} \\ &\le K \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+q+1}} M_p^q(\rho, f)\, d\rho. \end{aligned}$$

This proves the assertion made about $M_p^q(r, \frac{\partial f}{\partial z_k})$.

Lemma 4. *Let λ be a complex number with $|\lambda| < 1$ and $b > -1$, $a > b+1$, then*

$$\int_0^1 \frac{(1-\rho)^b}{|1-\lambda\rho|^a}\, d\rho \le \frac{K}{|1-\lambda|^{a-b-1}}.$$

The proof is a straightforward adaptation of ones given in [5, Lemma 2] and so is omitted.

We are now ready to give the

Proof of Theorem 1. Suppose $|\alpha| = 1$, we first prove the following

$$\int_0^1 (1-r)^{a+b} M_p^a\left(r, \frac{\partial f}{\partial z_k}\right) dr \le K \int_0^1 (1-r)^b M_p^a(r, f)\, dr. \tag{15}$$

If $a \le p$, taking $s > b$ in (11), then (15) immediately follows from Lemmas 3 and 4:

$$\begin{aligned} &\int_0^1 (1-r)^{a+b} M_p^a\left(r, \frac{\partial f}{\partial z_k}\right) dr \\ &\quad \le K \int_0^1 (1-\rho)^s M_p^a(\rho, f)\, d\rho \int_0^1 \frac{(1-r)^{a+b}}{(1-r\rho)^{s+a+1}}\, dr \\ &\quad \le K \int_0^1 (1-\rho)^b M_p^a(\rho, f)\, d\rho. \end{aligned}$$

If $a > p$, taking $q = p$ in (11) gives

$$M_p^a\left(r, \frac{\partial f}{\partial z_k}\right) \le K \left\{ \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+p+1}} M_p^p(\rho, f)\, d\rho \right\}^{a/p}.$$

Using Hölder's inequality with exponents a/p and $a/(a-p)$, we have

$$\begin{aligned} M_p^a\left(r, \frac{\partial f}{\partial z_k}\right) &\le K \left\{ \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+p+1}} M_p^a(\rho, f)\, d\rho \right\} \\ &\quad \times \left\{ \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+p+1}}\, d\rho \right\}^{(a-p)/p} \\ &\le K(1-r)^{p-a} \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+p+1}} M_p^a(\rho, f)\, d\rho. \end{aligned}$$

Taking $s > b$ and using Lemma 4 again yield

$$\begin{aligned}\int_0^1 &(1-r)^{a+b} M_p^a\left(r, \frac{\partial f}{\partial z_k}\right) dr \\ &\le K \int_0^1 (1-\rho)^s M_p^a(\rho, f)\, d\rho \int_0^1 \frac{(1-r)^{b+p}}{(1-r\rho)^{s+p+1}}\, d\rho \\ &\le K \int_0^1 (1-\rho)^b M_p^a(\rho, f)\, d\rho.\end{aligned}$$

This completes the proof of (15).

For $|\alpha| = 2$, using (15) twice gives

$$\begin{aligned}\int_0^1 (1-r)^{2a+b} M_p^a\left(r, \frac{\partial^2 f}{\partial z_j \partial z_k}\right) dr &\le K \int_0^1 (1-r)^{a+b} M_p^a\left(r, \frac{\partial f}{\partial z_k}\right) dr \\ &\le K \int_0^1 (1-r)^b M_p^a(r, f)\, dr.\end{aligned}$$

The general case can be proved by induction. Theorem 1 is proved.

3. The proof of Theorem 2

The following lemmas will be needed in the proof of Theorem 2.

Lemma 5 [4, p. 758]. *Let* $1 \le k < \infty$, $\mu > 0$, $\delta > 0, h : (0, 1) \to [0, \infty)$ *measurable, then*

$$\int_0^1 (1-r)^{k\mu-1} \left\{\int_0^r (r-t)^{\delta-1} h(t)\, dt\right\}^k dr \le K \int_0^1 (1-r)^{k\mu+k\delta-1} h^k(r)\, dr.$$

Lemma 6 [1]. *Let* $f \in H(B)$ *and if* $\sup_{0<r<1} \int_{\partial B} |f(r\zeta)|^p\, d\sigma(\zeta) = C^p$, $p > 0$, *then*

$$\int_{\partial B} \sup_{0<r<1} |f(r\zeta)|^p\, d\sigma(\zeta) \le A_p C^p.$$

Lemma 7. *Let* $f \in H(B)$ *and* $0 < r < 1$.

(a) *If* $p \ge 1$, *then*

$$M_p(r, f) \le K\left\{|f(0)| + r^{-1} \int_0^r M_p(s, |\nabla f|)\, ds\right\}. \tag{16}$$

(b) *If* $0 < p < 1$, *then*

$$M_p(r, f) \le K\left\{|f(0)| + r^{-1}\left(\int_0^r (r-s)^{p-1} M_p^p(s, |\nabla f|)\, ds\right)^{1/p}\right\}. \tag{17}$$

Proof. (a) Let $f \in H(B)$ have the homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} F_k(z).$$

We define the radial derivative Rf of f as follows:

$$(Rf)(z) = \sum_{k=0}^{\infty} kF_k(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z).$$

It is easy to see that $(Rf)(\lambda z) = \lambda f_z'(\lambda)$ for $z \in \overline{B}$ and $\lambda \in U$, where U denotes the unit disc and $f_z(\lambda) = f(\lambda z)$. Thus

$$f(z) = f(0) + \int_0^1 t^{-1}(Rf)(tz)\,dt \tag{18}$$

and

$$|f(z)|^p \le K\left\{|f(0)|^p + \left(\int_0^1 |(\nabla f)(tz)|\,dt\right)^p\right\} \tag{19}$$

since

$$t^{-1}|(Rf)(tz)| \le \sum_{k=1}^{n} |z_k| \left|\frac{\partial f}{\partial z_k}(tz)\right| \le |(\nabla f)(tz)|. \tag{20}$$

Applying Minkowski's inequality gives

$$\begin{aligned} M_p(r, f) &\le K\left\{|f(0)| + \int_0^1 M_p(tr, |\nabla f|)\,dt\right\} \\ &= K\left\{|f(0)| + r^{-1}\int_0^r M_p(s, |\nabla f|)\,ds\right\}. \end{aligned}$$

This is the desired inequality (16).

(b) If $0 < p < 1$, by (18) and (20),

$$|f(z)| \le |f(0)| + \int_0^1 |(\nabla f)(tz)|\,dt.$$

Denote $t_k = 1 - 2^{-k}$ and $H_k(z) = \sup\{|(\nabla f)(tz)| : t_{k-1} < t < t_k\}$, then

$$\int_0^1 |(\nabla f)(tz)|\,dt = \sum_{k=1}^{\infty} \int_{t_{k-1}}^{t_k} |(\nabla f)(tz)|\,dt \le \sum_{k=1}^{\infty} H_k(z) 2^{-k}.$$

Hence

$$|f(z)|^p \le |f(0)|^p + \sum_{k=1}^{\infty} H_k^p(z) 2^{-pk}.$$

By the monotonicity of the integral means $M_p(r, \frac{\partial f}{\partial z_k})$, there exists a constant K, depending only on p, such that

$$\sup\{M_p^p(rt, |\nabla f|) : t_{k-1} < t < t_k\} \le K M_p^p(rt_k, |\nabla f|). \tag{21}$$

Now Lemma 6 and (21) give

$$
\begin{aligned}
M_p^p(r, f) &\leq |f(0)|^p + \sum_{k=1}^{\infty} 2^{-pk} \sup\{M_p^p(rt, |\nabla f|) : t_{k-1} < t < t_k\} \\
&\leq |f(0)|^p + K \sum_{k=1}^{\infty} 2^{-pk} M_p^p(rt_k, |\nabla f|) \\
&\leq |f(0)|^p + K \sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} (1-t)^{p-1} M_p^p(rt, |\nabla f|)\, dt \\
&\leq |f(0)|^p + K \int_0^1 (1-t)^{p-1} M_p^p(rt, |\nabla f|)\, dt .
\end{aligned} \tag{22}
$$

Setting $rt = s$ in the right-hand integral of (22) gives (17). This completes the proof.

Lemma 8. *If $h(r)$ is a positive continuous nondecreasing function of r, and $\beta > 0$, $0 < c < q < \infty$, then for $0 < s < 1$,*

$$
\left\{ \int_0^1 (1-r)^{\beta q - 1} h^q(rs)\, dr \right\}^{1/q} \leq K \left\{ \int_0^1 (1-r)^{\beta c - 1} h^c(rs)\, dr \right\}^{1/c} .
$$

Using the same method as in the proof of Lemma 5 in [7] gives the lemma.

Proof of Theorem 2. We first assume that $|m| = 1$ and prove the following inequality:

(23)

$$
\int_0^1 (1-r)^b M_p^a(r, f)\, dr \leq K \left\{ |f(0)|^a + \sum_{k=1}^{n} \int_0^1 (1-r)^{a+b} M_p^a \left(r, \frac{\partial f}{\partial z_k} \right) dr \right\} .
$$

The proof will be divided into four steps.

Case 1. $p \geq 1$ and $a \geq 1$. By (16),

$$
r^a M_p^a(r, f) \leq K \left\{ |f(0)| + \int_0^r M_p(t, |\nabla f|)\, dt \right\}^a . \tag{24}
$$

Set $r = \rho^{a+1}$ in the left-hand integral of (23), use the monotonicity of means, the inequality $(1-\rho^{a+1})^b \leq K(1-\rho)^b$ and (24) to obtain

$$
\begin{aligned}
\int_0^1 (1-r)^b M_p^a(r, f)\, dr &\leq K \int_0^1 (1-\rho)^b M_p^a(\rho, f) \rho^a\, d\rho \\
&\leq K \left\{ |f(0)|^a + \int_0^1 (1-\rho)^b \left\{ \int_0^r M_p(t, |\nabla f|)\, dt \right\}^a d\rho \right\} .
\end{aligned}
$$

Taking $k = a$, $\mu = (1+b)/a$, $\delta = 1$, $h(t) = M_p(t, |\nabla f|)$ in Lemma 5 gives

$$
\int_0^1 (1-r)^b M_p^a(r, f)\, dr \leq K \left\{ |f(0)|^a + \int_0^1 (1-r)^{a+b} M_p^a(r, |\nabla f|)\, dr \right\} . \tag{25}
$$

Note that

$$M_p^a(r, |\nabla f|) \le K \sum_{k=1}^{n} M_p^a\left(r, \frac{\partial f}{\partial z_k}\right). \tag{26}$$

Thus (25) and (26) imply (23).

Case 2. $p \ge 1$ and $a < 1$. Apply the Minkowski's inequality to (19) and taking $c = a$, $q = \beta = 1$, and $h(r) = M_p(r, |\nabla f|)$ in Lemma 8, we obtain

$$\begin{aligned} M_p(r, f) &\le K\left\{|f(0)| + \int_0^1 M_p(tr, |\nabla f|)\,dt\right\} \\ &\le K\left\{|f(0)| + \left(\int_0^1 (1-t)^{a-1} M_p^a(tr, |\nabla f|)\,dt\right)^{1/a}\right\}, \end{aligned} \tag{27}$$

the substitution $s = tr$ gives

$$r^a M_p^a(r, f) \le K\left\{|f(0)|^a + \int_0^r (r-s)^{a-1} M_p^a(s, |\nabla f|)\,ds\right\}.$$

As in Case 1, we have

$$\begin{aligned} &\int_0^1 (1-r)^b M_p^a(r, f)\,dr \\ &\qquad \le K\left\{|f(0)|^a + \int_0^1 (1-\rho)^b \left\{\int_0^\rho (\rho - s)^{a-1} M_p^a(s, |\nabla f|)\,ds\right\} d\rho\right\}. \end{aligned}$$

Now taking $k = 1$, $\mu = 1 + b$, $\delta = a$ and $h(s) = M_p^a(s, |\nabla f|)$ in Lemma 5 gives (25) and (23) follows from (25) and (26).

Case 3. $0 < p < 1$ and $p \le a$. By (17), we have

$$r^a M_p^a(r, f) \le K\left\{|f(0)|^a + \left(\int_0^1 (r-s)^{p-1} M_p^p(s, |\nabla f|)\,ds\right)^{a/p}\right\}.$$

Let $r = \rho^{a+1}$, and

$$\begin{aligned} \int_0^1 (1-r)^b M_p^a(r, f)\,dr &\le K \int_0^1 (1-\rho)^b M_p^a(\rho, f)\rho^a\,d\rho \\ &\le K\left\{|f(0)|^a + \int_0^1 (1-\rho)^b \left(\int_0^\rho (\rho-s)^{p-1} M_p^p(s, |\nabla f|)\,ds\right)^{a/p} d\rho\right\}. \end{aligned}$$

Now taking $k = a/p$, $\mu = (1+b)p/a$, $\delta = p$ and $h(s) = M_p^p(s, |\nabla f|)$ in Lemma 5 gives (25) and (23) follows from (25) and (26).

Case 4. $0 < p < 1$ and $p > a$. By the inequality (22) and taking $h(r) = M_p(r, |\nabla f|)$ in Lemma 8, we get

$$M_p(r, f) \le K\left\{|f(0)| + \left(\int_0^1 (1-t)^{p-1} M_p^p(rt, |\nabla f|)\,dt\right)^{1/p}\right\}$$
$$\le K\left\{|f(0)| + \left(\int_0^1 (1-t)^{a-1} M_p^a(rt, |\nabla f|)\,dt\right)^{1/a}\right\}.$$

This is just the inequality (27) and the rest of the proof follows as in the Case 2. The proof of (23) is completed.

For $|m| = 2$, we apply (23) to $\frac{\partial f}{\partial z_k}$ to obtain

$$\text{(28)}\quad \int_0^1 (1-r)^b M_p^a\left(r, \frac{\partial f}{\partial z_k}\right) dr$$
$$\le K\left\{\left|\frac{\partial f}{\partial z_k}(0)\right|^a + \sum_{j=1}^n \int_0^1 (1-r)^{a+b} M_p^a\left(r, \frac{\partial^2 f}{\partial z_k \partial z_j}\right) dr\right\}.$$

Substituting (28) into (23) gives

$$\int_0^1 (1-r)^b M_p^a(r, f)\,dr$$
$$\le K\left\{\sum_{|\alpha|\le 1} |(D^\alpha)(0)|^a + \sum_{k,j=1}^n \int_0^1 (1-r)^{2a+b} M_p^a\left(r, \frac{\partial^2 f}{\partial z_j \partial z_k}\right) dr\right\}$$
$$\le K\left\{\sum_{|\alpha|\le 1} |(D^\alpha f)(0)|^a + \sum_{|\alpha|=2} \int_0^1 (1-r)^{2a+b} M_p^a(r, D^\alpha f)\,dr\right\}.$$

As in Theorem 1, the general case can be proved by induction. The proof of Theorem 2 is completed.

4. An application of Theorems 1 and 2

In [9], the main tools for proving Theorem A are

(1) The boundedness of a family of projections from $L^p(d\nu)$ onto $L^p(d\nu) \cap H(B)$ for all $p \in [1, \infty)$.

(2) The solutions of the generalized Gleason problem for the Bergman spaces.

Because the boundedness of the projections can be used only in the case $p \ge 1$, the method used in [9] cannot deal with the case $0 < p < 1$. Using Theorems 1 and 2, we can prove that Theorem A holds for all $p \in (0, \infty)$.

Let $f \in H(B)$ and $0 < p < \infty$. As in [9], we define

$$\|f\|_{m,p} = \sum_{|\alpha|\le m-1} |(D^\alpha f)(0)| + \sum_{|\alpha|=m} \|T_\alpha f\|_p,$$

where $(T_\alpha f)(z) = (1-|z|^2)^{|\alpha|}(D^\alpha f)(z)$.

Theorem 3. *Let m be a positive integer and $f \in H(B)$, then $f \in L^p(d\nu)$, $0 < p < \infty$, if and only if all the functions $(1-|z|^2)^m (D^\alpha f)(z)$ with $|\alpha| = m$ are in $L^p(d\nu)$. Moreover, $\| \|_p$ and $\| \|_{m,p}$ are equivalent norms on $L^p(d\nu) \cap H(B)$.*

Before proving the theorem, we first prove a simple lemma.

Lemma 9. *Let $f \in H(B)$ and $0 < p < \infty$, then for any $\alpha = (\alpha_1, \dots, \alpha_n)$,*

(a) *If $T_\alpha f \in L^p(d\nu)$, then $\int_0^1 M_p^p(r, T_\alpha f)\,dr \le K\|T_\alpha f\|_p^p$.*

(b) *If $\int_0^1 M_p^p(r, T_\alpha f)\,dr < \infty$, then $T_\alpha f \in L^p(d\nu)$ and*

$$\|T_\alpha f\|_p^p \le K \int_0^1 M_p^p(r, T_\alpha f)\,dr.$$

Proof. Since

$$\begin{aligned}\int_B |(T_\alpha f)(z)|^p\,d\nu(z) &= 2n\int_0^1 r^{2n-1}\,dr \int_{\partial B} |(T_\alpha f)(r\zeta)|^p\,d\sigma(\zeta) \\ &\le 2n \int_0^1 M_p^p(r, T_\alpha f)\,dr,\end{aligned} \tag{29}$$

(b) follows. On the other hand, by the monotonicity of the integral mean of holomorphic function, we have

$$\begin{aligned}\int_0^1 M_p^p(r, T_\alpha f)\,dr &= \int_0^1 (1-r^2)^{|\alpha|p} M_p^p(r, D^\alpha f)\,dr \\ &= 2n\int_0^1 (1-\rho^{4n})^{|\alpha|p} M_p^p(\rho^{2n}, D^\alpha f)\rho^{2n-1}\,d\rho \\ &\le 2nK \int_0^1 (1-\rho^2)^{|\alpha|p} M_p^p(\rho, D^\alpha f)\rho^{2n-1}\,d\rho \\ &= K\int_B |(T_\alpha f)(z)|^p\,d\nu(z) = K\|T_\alpha f\|_p^p,\end{aligned}$$

which proves the assertion (a).

Proof of Theorem 3. We first assume that $f \in L^p(d\nu)$. Take $a = p$, $b = 0$ in Theorem 1 and use Lemma 9, we obtain

$$\begin{aligned}\|T_\alpha f\|_p^p &\le K\int_0^1 M_p^p(r, T_\alpha f)\,dr = K\int_0^1 (1-r^2)^{|\alpha|p} M_p^p(r, D^\alpha f)\,dr \\ &\le K\int_0^1 M_p^p(r, f)\,dr \le K\|f\|_p^p.\end{aligned}$$

This proves that $T_\alpha f \in L^p(B, d\nu)$ for any $\alpha = (\alpha_1, \dots, \alpha_n)$ and

$$\|T_\alpha f\|_p \le K\|f\|_p. \tag{30}$$

On the other hand, replace f by $D^\alpha f$ and set $s = |\alpha|p$, $z = 0$ in (7), then using the inequality (30), we get

$$|(\nabla D^\alpha f)(0)|^p \le K\int_B |(D^\alpha f)(w)|^p (1-|w|^2)^{|\alpha|p}\,d\nu(w) = K\|T_\alpha f\|_p^p \le K\|f\|_p^p.$$

This shows that for any $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$|(D^\alpha f)(0)| \leq K\|f\|_p. \tag{31}$$

Combining (30) and (31) gives $\|f\|_{m,p} \leq K\|f\|_p$.

Next, suppose that $T_\alpha f \in L^p(B, d\nu)$ for all α with $|\alpha| = m$. Take $a = p$, $b = 0$ in Theorem 2 and use Lemma 9, we have

$$\begin{aligned}
\|f\|_p^p &\leq K \int_0^1 M_p^p(r, f)\, dr \\
&\leq K \left\{ \sum_{|\alpha| \leq m-1} |(D^\alpha f)(0)|^p + \sum_{|\alpha|=m} \int_0^1 (1-r)^{pm} M_p^p(r, D^\alpha f)\, dr \right\} \\
&\leq K \left\{ \sum_{|\alpha| \leq m-1} |(D^\alpha f)(0)|^p + \sum_{|\alpha|=m} \int_0^1 M_p^p(r, T_\alpha f)\, dr \right\} \\
&\leq K \left\{ \sum_{|\alpha| \leq m-1} |(D^\alpha f)(0)|^p + \sum_{|\alpha|=m} \|T_\alpha f\|_p^p \right\}.
\end{aligned}$$

This implies that

$$\|f\|_p \leq K \left\{ \sum_{|\alpha| \leq m-1} |(D^\alpha f)(0)| + \sum_{|\alpha|=m} \|T_\alpha f\|_p \right\} = K\|f\|_{m,p}. \tag{32}$$

Theorem 3 is proved.

If we define

$$d\nu_b(z) = (1 - |z|^2)^b\, d\nu(z), \qquad b > -1; \qquad \|f\|_{p,b}^p = \int_B |f(z)|^p\, d\nu_b(z)$$

and

$$\|f\|_{m,p,b} = \sum_{|\alpha| \leq m-1} |(D^\alpha f)(0)| + \sum_{|\alpha|=m} \|T_\alpha f\|_{p,b}, \tag{33}$$

then it is not hard to see from the above proof that Theorem 3 can be generalized as follows:

Theorem 4. *Let $f \in H(B)$ and $0 < p < \infty$, $m \geq 1$ be an integer, then $f \in L^p(B, d\nu_b)$ if and only if the functions $(1 - |z|^2)^m (D^\alpha f)(z)$ with $|\alpha| = m$ are in $L^p(B, d\nu_b)$. Moreover $\|\ \|_{p,b}$ and $\|\ \|_{m,p,b}$ are equivalent norms on $L^p(B, d\nu_b)$.*

In [7], we proved the following

Theorem D. *Let $f \in H(B)$, $0 < p \leq \infty$, $-1 < b < \infty$ and $0 < a < \infty$, then for any $\beta > 0$,*

(a) $\int_0^1 (1-r)^{a\beta+b} M_p^a(r, f^{[\beta]})\, dr \leq K \int_0^1 (1-r)^b M_p^a(r, f)\, dr$.

(b) $\int_0^1 (1-r)^b M_p^a(r, f)\, dr \leq K \int_0^1 (1-r)^{a\beta+b} M_p^a(r, f^{[\beta]})\, dr$.

Here $f^{[\beta]}$ denotes the βth fractional derivative of f (see [7, p. 162]). Using the same method as in the proof of Theorems 3 and 4, we have

Theorem 5. *Let* $f \in H(B)$, $0 < p < \infty$, $\beta > 0$, $b > -1$ *and denote* $(H_\beta f)(z) = (1-|z|^2)^\beta f^{[\beta]}(z)$.

(a) *If* $f \in L^p_a(B, d\nu_b)$, *then for any* $\beta > 0$, $H_\beta f \in L^p(B, d\nu_b)$ *and*

$$\|H_\beta f\|_{p,b} \le K\|f\|_{p,b}.$$

(b) *If for some* $\beta > 0$, $H_\beta f \in L^p(B, d\nu_b)$, *then* $f \in L^p_a(B, d\nu_b)$ *and*

$$\|f\|_{p,b} \le K\|H_\beta f\|_{p,b}.$$

Combining Theorems 4 and 5, we get

Theorem 6. *Let* $f \in H(B)$, $0 < p < \infty$, $\beta > 0$, $b > -1$ *and* $\alpha = (\alpha_1, \dots, \alpha_n)$.

(a) *If for some* $\beta > 0$, $H_\beta f \in L^p(B, d\nu_b)$, *then* $T_\alpha f \in L^p(B, d\nu_b)$ *for any* α, *and*

$$\|T_\alpha f\|_{p,b} \le K\|H_\beta f\|_{p,b}, \qquad |\alpha| = m.$$

(b) *If for all* α *with* $|\alpha| = m$, $T_\alpha f \in L^p(B, d\nu_b)$, *then* $H_\beta f \in L^p(B, d\nu_b)$ *for any* $\beta > 0$, *and*

$$\|H_\beta f\|_{p,b} \le K\|T_\alpha f\|_{p,b}.$$

5. Pluriharmonic conjugates

In this section, we will apply the approach used in the preceding sections to discuss the problem on pluriharmonic conjugates. The main theorem in this section is Theorem 7. The following lemma will be needed in the proof of Theorem 7.

Lemma 10. *Let* u *be pluriharmonic in* B *and* $0 < p < 1$, *then*

(a) $$|u(a)|^p \le K(1-|a|^2)^{-(n+1)} \int_B |u(z)|^p \, d\nu(z), \qquad a \in B.$$

(b) $$|u(\tfrac{1}{2}\zeta)| \le K \int_B |u(z)|^p (1-|z|^2)^s \, d\nu(z), \qquad s \ge 0,\ \zeta \in \partial B.$$

Proof. Using the same method as in [5, p. 64] gives

(34) $$|u(0)|^p \le \int_B |u(z)|^p \, d\nu(z).$$

Thus, (a) and (b) follow from (34) by the similar approach used in the proof of Lemma 1.

Proof of Theorem 7. Since

$$f(z) = u(z) + iv(z) = \sum_{\alpha \ge 0} \frac{(D^\alpha f)(0)}{\alpha!} z^\alpha.$$

It is not hard to see that

$$2\int_{\partial B} u(r\zeta)\overline{\zeta}^\alpha \, d\sigma(\zeta) = \frac{(D^\alpha f)(0)}{\alpha!}\omega_\alpha r^{|\alpha|}$$

and

$$\left|\frac{\partial f}{\partial z_k}(0)\right| \le (2n)^p \sup_{\zeta\in\partial B} |u(\tfrac{1}{2}\zeta)| \le K\int_B |u(z)|^p(1-|z|^2)^s\,d\nu(z) \tag{35}$$

for $k = 1, \ldots, n$, by Lemma 10 (b). Now apply the same approach as in the proof of Lemma 2 to (35), we obtain

$$|(\nabla f)(z)|^p \le K\int_B |u(w)|^p \frac{(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+p+1}}\,d\nu(w). \tag{36}$$

By the inequality [5, Lemma 1]: $|1-t\lambda|^{-1} \le 2|1-\lambda|^{-1}$, $0 < t < 1$, $|\lambda| < 1$, (36) can be written as

$$|(\nabla f)(tz)|^p \le K\int_B |u(w)|^p \frac{(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+p+1}}\,d\nu(w), \qquad 0 < t < 1.$$

Let $g(z) = \sup_{0<r<1} |(\nabla f)(rz)|$, then

$$|g(z)|^p \le K\int_B |u(w)|^p \frac{(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+p+1}}\,d\nu(w) \tag{37}$$

and

$$\int_0^1 |(\nabla f)(tz)|\,dt = \int_0^{1/r} |(\nabla f)(r\rho z)|r\,d\rho \le \int_0^{1/r} g(\rho z)\,d\rho.$$

Letting $r \to 1$ in the above inequality yields

$$\int_0^1 |(\nabla f)(tz)|\,dt \le \int_0^1 g(\rho z)\,d\rho. \tag{38}$$

Since $g(\rho z)$ is a positive nondecreasing function of ρ, taking $h(\rho) = g(\rho z)$ and $\beta = q = 1$, $c = p$ in Lemma 8 implies

$$\int_0^1 g(\rho z)\,d\rho \le K\left\{\int_0^1 (1-\rho)^{p-1} g^p(\rho z)\,d\rho\right\}^{1/p}. \tag{39}$$

Combining (18), (20), (38), (39), (37) and Lemma 4, we have

(40)

$$\begin{aligned}
|f(z)|^p &\le |f(0)|^p + K\int_0^1 (1-\rho)^{p-1}\left\{\int_B |u(w)|^p \frac{(1-|w|^2)^s}{|1-\rho\langle z, w\rangle|^{n+s+p+1}}\,d\nu(w)\right\}d\rho \\
&\le |f(0)|^p + K\int_B (1-|w|^2)^s|u(w)|^p\left\{\int_0^1 \frac{(1-\rho)^{p-1}\,d\rho}{|1-\rho\langle z, w\rangle|^{n+s+p+1}}\right\}d\nu(w) \\
&\le |f(0)|^p + K\int_B |u(w)|^p \frac{(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+1}}\,d\nu(w).
\end{aligned}$$

Note that if we use $u_{1/2}(w) = u(\frac{1}{2}w)$ in place of u in Lemma 10(a) and take $a = 0$, we obtain

$$|u(0)|^p \le K\int_B |u(\tfrac{1}{2}w)|^p\,d\nu(w) = 2^{2n}K\int_{B/2} |u(w)|^p\,d\nu(w).$$

On the other hand, for any $s \geq 0$,

$$\begin{aligned}\int_B |u(w)|^p(1-|w|^2)^s\,d\nu(w) &= 2n\int_0^1 \rho^{2n-1}(1-\rho^2)^s\left\{\int_{\partial B}|u(\rho\zeta)|^p\,d\sigma(\zeta)\right\}d\rho\\ &\geq K\int_0^{1/2}\rho^{2n-1}\left\{\int_{\partial B}|u(\rho\zeta)|^p\,d\sigma(\zeta)\right\}d\rho\\ &\geq K\int_{B/2}|u(w)|^p\,d\nu(w).\end{aligned}$$

Hence

$$\begin{aligned}|u(0)|^p &\leq K\int_B |u(w)|^p(1-|w|^2)^s\,d\nu(w)\\ &\leq K\int_B |u(w)|^p\frac{(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+1}}\,d\nu(w)\end{aligned}\tag{41}$$

for any $s \geq 0$ and $z \in B$. Now (40) and (41) gives

$$\begin{aligned}|f(z)|^p &\leq |u(0)|^p + |v(0)|^p + K\int_B |u(w)|^p\frac{(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+1}}\,d\nu(w)\\ &\leq |v(0)|^p + K\int_B |u(w)|^p\frac{(1-|w|^2)^s}{|1-\langle z, w\rangle|^{n+s+1}}\,d\nu(w).\end{aligned}$$

This completes the proof.

Applying the similar method as in the proof of Lemma 3, we can prove the following result from Theorem 7.

Theorem 8. *Let* $f = u + iv$ *be holomorphic in* B *with* $v(0) = 0$. *If* $0 < q < 1$, $p \geq q$ *and* $s \geq 0$, *then*

$$M_p^q(r, f) \leq K\int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+1}}M_p^q(\rho, u)\,d\rho.$$

The details of the proof are omitted here.

Theorem 8 will lead to the generalization of Theorems B and C in the case $0 < p < 1$.

Theorem 9. *Let* $f = u + iv \in H(B)$ *and* $0 < p < \infty$, *if*

$$M_p(r, u) = O((1-r)^{-\alpha}), \qquad \alpha > 0,$$

then

$$M_p(r, v) = O((1-r)^{-\alpha}).$$

Proof. We only need prove the case $0 < p < 1$. Without loss of generality, we assume $v(0) = 0$. Taking $p = q$ in Theorem 8 and using Lemma 4 yield

$$M_p^p(r, f) = O\left(\int_0^1 \frac{(1-\rho)^{s-p\alpha}}{(1-r\rho)^{s+1}}\,d\rho\right) = O((1-r)^{-p\alpha}).$$

The theorem is proved.

Theorem 10. *Let $f \in u + iv \in H(B)$ with $v(0) = 0$ and $0 < p \leq \infty$, $a > 0$, $b > -1$, then*

$$\int_0^1 (1-r)^b M_p^a(r, v)\,dr \leq K \int_0^1 (1-r)^b M_p^a(r, u)\,dr.$$

Proof. It is known that the theorem holds for $p \geq 1$. We assume that $0 < p < 1$. The proof will be divided into three steps.

Case 1. $a = p$. Take $s > b$ in Theorem 8 and use Lemma 4, we have

$$\begin{aligned}\int_0^1 (1-r)^b M_p^p(r, f)\,dr &\leq K \int_0^1 (1-\rho)^s M_p^p(\rho, u) \left\{ \int_0^1 \frac{(1-r)^b}{(1-r\rho)^{s+1}}\,dr \right\} d\rho \\ &\leq K \int_0^1 (1-\rho)^b M_p^p(\rho, u)\,d\rho.\end{aligned}$$

Case 2. $a > p$. Taking $q = p$ in Theorem 8 we obtain

$$M_p^a(r, f) \leq K \left\{ \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+1}} M_p^p(\rho, u)\,d\rho \right\}^{a/p}.$$

Using Hölder's inequality with exponents a/p and $a/(a-p)$ gives

$$\begin{aligned}M_p^a(r, f) \leq K &\left\{ \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+1-\varepsilon p/a}} M_p^a(\rho, u)\,d\rho \right\} \\ &\times \left\{ \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+1+a\varepsilon/(a-p)}}\,d\rho \right\}^{(a-p)/p},\end{aligned}$$

where ε is a small positive number. Taking $s > b$ and using Lemma 4 twice imply

$$\begin{aligned}\int_0^1 (1-r)^b M_p^a(r, f)\,dr &\leq K \int_0^1 (1-\rho)^s M_p^a(\rho, u) \left\{ \int_0^1 \frac{(1-r)^{b-\varepsilon a/p}}{(1-r\rho)^{s+1-\varepsilon p/a}}\,dr \right\} d\rho \\ &\leq K \int_0^1 (1-\rho)^b M_p^a(\rho, u)\,d\rho.\end{aligned}$$

Case 3. $a < p$. By Theorem 8,

$$M_p^a(r, f) \leq K \int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+1}} M_p^a(\rho, u)\,d\rho.$$

Thus the desired inequality follows from Lemma 4 with $s > b$. The theorem is proved.

As an application of Theorem 10, we have

Theorem 11. *Let $f = u + iv \in H(B)$ with $v(0) = 0$. If $u \in h\wedge(\alpha, p, q)$, then $v \in h\wedge(\alpha, p, q)$ for $0 < p \leq \infty$, $0 < q \leq \infty$ and $\alpha > 0$, moreover,*

$$N_{p,q,\alpha}(v) \leq K N_{p,q,\alpha}(u).$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, ANHUI, 230026, PEOPLE'S REPUBLIC OF CHINA